Written trial exam, Nov 2023

Suggested solution

Q1

$(\neg(P \vee Q)) \Rightarrow P$ and $P \vee Q$ are logically equivivant if their truth tables are identical.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$(\neg(P \vee Q)) \Rightarrow P$
T	T	T	F	T
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F
		↑		↑

They are logically equivalent.

Q2

Given polynomial: $P(z)=z^3+27$.

Guessing a root $z_0=-3$. Checking: $(-3)^3+27=-27+27=0$, ok.

Using the polynomial division algorithm to reduce to a lower-degree polynomial:

$$
\begin{array}{l|llll}
z-(-3) & z^3 & & +27 & \lfloor\, z^2-3z+9 \leftarrow Q(z) \\
 & z^3+3z^2 & & & \\ \hline
 & -3z^2 & & +27 & \\
 & -3z^2 & -9z & & \\ \hline
 & & 9z & +27 & \\
 & & 9z & +27 & \\ \hline
 & & & 0 &
\end{array}
$$

So, a factorization of the polynomial is

$P(z)=(z-(-3))\,Q(z) = (z+3)(z^2-3z+9)$

Discriminant of $Q(z)$: $\text{Discr} = (-3)^2-4\cdot1\cdot9 = 9-36 = -27$

Roots of $Q(z)$: $z = \dfrac{-(-3)\pm\sqrt{-27}}{2\cdot1} = \dfrac{3}{2} \pm i\dfrac{3\sqrt{3}}{2}$

All roots of the given polynomial are hence:

-3, $\dfrac{3}{2}+i\dfrac{3\sqrt{3}}{2}$, $\dfrac{3}{2}-i\dfrac{3\sqrt{3}}{2}$

Q3

Given recursion for $(s_1, s_2, s_3, \dots)$:

$$s_n = \begin{cases} 0 & \text{if } n=1 \\ 2s_{n-1}+2 & \text{if } n\geq 2 \end{cases}$$

a) $s_1 = 0$, $s_2 = 2s_1+2 = 2\cdot 0+2 = 2$

$s_3 = 2\cdot s_2+2 = 2\cdot 2+2 = 6$

b) It is claimed for all $n\in\mathbb{Z}_{\geq 1}$ that:

$$s_n = 2^n - 2 .$$

We will show this using induction on n:

Base case: For $n=1$ we see $s_1 = 2^1-2 = 0$, so the expression holds for $n=1$.

Induction step: Assuming true for $n-1$, so

$$s_{n-1} = 2^{n-1}-2 .$$

We rewrite:

$$s_n = 2s_{n-1}+2 = 2(2^{n-1}-2)+2$$
$$= 2^n - 4 + 2 = 2^n - 2$$

so, if the expression is true for s_{n-1}, it is also true for s_n, for $n>1$

We conclude that the expression is true for $n\geq 1$

Q4

Given system of linear equations:

$$\begin{cases} x_1 + x_2 + 2x_3 = 1 \\ x_2 - 4x_3 + 4x_4 = 5 \\ 3x_1 + 3x_2 + 2x_3 + 4x_4 = -1 \end{cases} \quad \text{over } \mathbb{R}.$$

a) Rewriting:

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -4 & 4 \\ 3 & 3 & 2 & 4 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}}_{\underline{b}}$$

$\underline{\underline{A}}$ is the coefficient matrix, and $\underline{b}$ the right-hand side.

Since $\underline{b} \neq \underline{0}$, the system is <u>inhomogeneous</u>.

b) If $\underline{v} \in \mathbb{R}^4$ and $\underline{w} \in \mathbb{R}^4$ are two different solutions to the inhomogeneous system, then

$$\underline{\underline{A}}\,\underline{v} = \underline{b} \quad \text{and} \quad \underline{\underline{A}}\,\underline{w} = \underline{b}.$$

Since the system is linear, then the difference between them, $\underline{v} - \underline{w}$, gives:

$$\underline{\underline{A}}(\underline{v} - \underline{w}) = \underline{\underline{A}}\,\underline{v} - \underline{\underline{A}}\,\underline{w} = \underline{b} - \underline{b} = \underline{0}.$$

$\underline{v} - \underline{w}$ is hence a solution to the corresponding homogeneous system, but NOT to the given inhomogeneous system.

Q5

Given: $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 1 \\ 3 & 0 & 8 \end{bmatrix} \in \mathbb{C}^{3\times 3}$

a) Characteristic polynomial:

$$P_{\underline{\underline{A}}}(z) = \det(\underline{\underline{A}} - z\,\underline{\underline{I}}_3) = \det\left(\begin{bmatrix} 1-z & 0 & 0 \\ 4 & -z & 1 \\ 3 & 0 & 8-z \end{bmatrix}\right)$$

$$= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det(\underline{\underline{A}}(1;j)) \longleftarrow$$

Using the expansion method, expanding by row $i=1$.

$$= (-1)^{1+1}(1-z)\det\left(\begin{bmatrix} -z & 1 \\ 0 & 8-z \end{bmatrix}\right) + (-1)^{1+2}\cdot\cancel{0\cdot\det(\underline{\underline{A}}(1;2))} + (-1)^{1+3}\cdot\cancel{0\cdot\det(\underline{\underline{A}}(1;3))}$$

$$= (1-z)(-z(8-z) - 0\cdot 1)$$

$$= (1-z)(-8z + z^2)$$

$$= -z^3 + 8z^2 + z^2 - 8z$$

$$= -z^3 + 9z^2 - 8z$$

b)

Given eigenvectors of $\underline{\underline{A}}$: $\underline{v}_1 = \begin{bmatrix} -7 \\ -25 \\ 3 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}$.

Considering $\underline{\underline{A}}$ as a mapping matrix of a map $\mathbb{R}^3 \to \mathbb{R}^3$, we map the eigenvectors:

$$\underline{\underline{A}}\,\underline{v}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 1 \\ 3 & 0 & 8 \end{bmatrix}\begin{bmatrix} -7 \\ -25 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ -28+3 \\ -21+24 \end{bmatrix} = \begin{bmatrix} -7 \\ -25 \\ 3 \end{bmatrix} = 1 \cdot \underline{v}_1$$

$$\underline{\underline{A}}\,\underline{v}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 1 \\ 3 & 0 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 64 \end{bmatrix} = 8 \cdot \underline{v}_2$$

so, $\underline{v}_1$ has eigenvalue $\lambda_1 = 1$ and
$\underline{v}_2$ has eigenvalue $\lambda_2 = 8$.

Q6

Given real Vector space V, where $\dim(V) = 2$, and change-of-basis matrix changing from basis β to basis γ:

$$_{\gamma}[\mathrm{id}]_{\beta} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

The opposite change-of-basis matrix from γ to β basis is its inverse $_{\beta}[\mathrm{id}]_{\gamma} = (_{\gamma}[\mathrm{id}]_{\beta})^{-1}$.

Finding this inverse:

$$\left[\, _{\gamma}[\mathrm{id}]_{\beta} \;\middle|\; \underline{\underline{I}}_2 \,\right] = \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 : R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right] \xrightarrow{R_2 : R_2 \cdot (-1)}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 0 & 1 \\ -1 & 1 \end{matrix}}_{(_{\gamma}[\mathrm{id}]_{\beta})^{-1}}$$

we read the inverse to be:

$$(_{\gamma}[\mathrm{id}]_{\beta})^{-1} = {_{\beta}[\mathrm{id}]_{\gamma}} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Q7

Given 2nd-order differential equation:

$$f''(t) - 3f'(t) + 2f(t) = 2t$$

a) Given function $f_0(t) = t + \frac{3}{2}$. Inserting:

$$(t+\tfrac{3}{2})'' - 3(t+\tfrac{3}{2})' + 2(t+\tfrac{3}{2}) = 2t$$

$$\Updownarrow \quad 0 - \cancel{3} \cdot 1 + 2t + \cancel{3} = 2t$$

$$\Updownarrow \quad 2t = 2t$$

, so $f_0(t)$ is a particular solution

b) The diff. equation is inhomogeneous since $q(t) = 2t \neq 0$ where $q(t)$ is a forcing function as defined in Definition 12.23.

The inhomogeneous solution set is, according to Theorem 12.26, the sum of the solution set to the corresponding homogeneous diff. equation and a particular solution.

The general solution to the homogeneous diff. equation where $q(t) = 0$, $f''(t) - 3f'(t) + 2f(t) = 0$, is, according to equation (12-14) in Note 12:

$$f_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad c_1, c_2 \in \mathbb{R}$$

where λ_1 and λ_2 are roots in the polynomial:

$$z^2 - 3z + 2 = 0$$

Discriminant: $\text{Discr} = (-3)^2 - 4 \cdot 1 \cdot 2 = 9 - 8 = 1$

The roots are $\lambda = \frac{-(-3) \pm \sqrt{1}}{2 \cdot 1} = \frac{3 \pm 1}{2} \Leftrightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 1 \end{cases}$

Hence $f_h(t) = c_1 e^{2t} + c_2 e^{t}$ and the general solution to the inhomogeneous equation is:

$$f(t) = \underbrace{c_1 e^{2t} + c_2 e^{t}}_{f_h(t)} + \underbrace{t + \tfrac{3}{2}}_{f_0(t)}, \quad c_1, c_2 \in \mathbb{R}.$$