

Danmarks Tekniske Universitet

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Skriftlig eksamen, d. 8. december 2024

Kursusnavn: Matematik 1a (Polyteknisk grundlag)

Kursusnr. 01001\01003

Eksamensvarighed: 2 timer

Hjælpemidler: Ingen elektroniske hjælpemidler

“Vægtning”: Alle spørgsmål i denne eksamen vægtes ens. Denne del af eksamenen tæller 50% af hele eksamenen.

Yderligere information: Spørgsmålene er stillet først på dansk og dernæst på engelsk. Alle svar skal være motiverede, og mellemregninger skal angives i passende omfang.

The Technical University of Denmark

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Written exam, the 8th of December 2024

Course name: Mathematics 1a (Polytechnical foundation)

Course no. 01001\01003

Exam duration: 2 hours

Aid: No electronic aid

“Weighting”: All questions in this exam are weighted equally. This part of the exam counts for 50% of the whole exam.

Additional information: The questions are posed first in Danish and then in English. All answers have to be motivated and intermediate steps need to be given to a reasonable extent.

Opgave 1

- a) Beregn sandhedstabellen for følgende logiske udsagn:

$$(\neg(Q \Leftrightarrow P)) \wedge (\neg Q).$$

- b) Er det logiske udsagn $(\neg(Q \Leftrightarrow P)) \wedge (\neg Q)$ logisk ækvivalent med det logiske udsagn $P \wedge (\neg Q)$?

Opgave 2

Givet polynomiet $p(Z) = 2Z^3 - 2Z^2 - 8Z - 12$ i $\mathbb{C}[Z]$. Det oplyses at 3 er rod i dette polynomium.

- a) Skriv polynomiet $p(Z)$ som produkt af et polynomium af grad et og et polynomium af grad to.
- b) Find samtlige rødder i $p(Z)$ i \mathbb{C} . Rødderne skal angives på rektangulær form.

Opgave 3

Betragt for $a \in \mathbb{C}$ følgende matrix:

$$\mathbf{A} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \in \mathbb{C}^{2 \times 2}.$$

Besvar følgende spørgsmål:

- a) Beregn \mathbf{A}^2 og \mathbf{A}^3 .
- b) Vis ved hjælp af induktion efter n at $\mathbf{A}^n = \begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix}$ for alle $n \in \mathbb{Z}_{\geq 2}$.

Opgave 4

Der gives følgende matrix i $\mathbb{R}^{3 \times 3}$:

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & -3 \\ 2 & 1 & 4 \\ 2 & -1 & 8 \end{bmatrix}.$$

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- a) Find en basis for \mathbf{B} 's kerne.
- b) Find en basis for \mathbf{B} 's søjlerum.
- c) Betragt den lineære afbildning $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ givet ved $L(\mathbf{v}) = \mathbf{B} \cdot \mathbf{v}$. Er den lineære afbildning L surjektiv?

Opgave 5

Lad $V = \mathbb{R}^3$ og lad

$$\beta = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

være den ordnede standardbasis for V . Ydermere vælges følgende ordnede basis for V :

$$\gamma = \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right).$$

- a) Angiv basisskiftmatricen ${}_{\beta}[\text{id}_{\mathbb{R}^3}]_{\gamma}$.
- b) Beregn basisskiftmatricen ${}_{\gamma}[\text{id}_{\mathbb{R}^3}]_{\beta}$.

Opgave 6

Givet følgende system af førsteordens ODEs:

$$\begin{cases} f_1'(t) = f_1(t) + 2f_2(t) \\ f_2'(t) = 2f_1(t) + f_2(t) \end{cases}$$

- a) Er det givne differentiaalligningssystem homogent eller inhomogent?
- b) Beregn differentiaalligningssystemets fuldstændige reelle løsning.
- c) Bestem en løsning til det givne differentiaalligningssystem som opfylder begyndelsesbetingelserne $f_1(0) = 3$ og $f_2(0) = 5$.

EKSAMEN SLUT

Question 1

- a) Determine the truth table of the following logical proposition:

$$(\neg(Q \Leftrightarrow P)) \wedge (\neg Q).$$

- b) Is the logical proposition $(\neg(Q \Leftrightarrow P)) \wedge (\neg Q)$ logically equivalent to the logical proposition $P \wedge (\neg Q)$?

Question 2

Given is the polynomial $p(Z) = 2Z^3 - 2Z^2 - 8Z - 12$ in $\mathbb{C}[Z]$. It is given that 3 is a root of this polynomial.

- a) Write the given polynomial $p(Z)$ as a product of a polynomial of degree one and a polynomial of degree two.
- b) Find all roots of $p(Z)$ in \mathbb{C} . The roots should be given in rectangular form.

Question 3

Let $a \in \mathbb{C}$ be given and define the matrix

$$\mathbf{A} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \in \mathbb{C}^{2 \times 2}.$$

Now answer the following questions:

- a) Compute \mathbf{A}^2 and \mathbf{A}^3 .
- b) Show using induction on n that $\mathbf{A}^n = \begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix}$ for all $n \in \mathbb{Z}_{\geq 2}$.

Question 4

The following matrix in $\mathbb{R}^{3 \times 3}$ is given:

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & -3 \\ 2 & 1 & 4 \\ 2 & -1 & 8 \end{bmatrix}.$$

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- a) Compute a basis for the kernel of the given matrix \mathbf{B} .
- b) Compute a basis for the column space of the given matrix \mathbf{B} .
- c) Consider the linear map $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{B} \cdot \mathbf{v}$. Is the linear map L surjective?

Question 5

Let $V = \mathbb{R}^3$. Denote by

$$\beta = \left(\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] \right)$$

the ordered standard basis for V . The following ordered basis for V is chosen:

$$\gamma = \left(\left[\begin{array}{c} 2 \\ 1 \\ 0 \end{array} \right], \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 0 \\ 2 \end{array} \right] \right).$$

- a) Compute the change of basis matrix ${}_{\beta}[\text{id}_{\mathbb{R}^3}]_{\gamma}$.
- b) Compute the change of basis matrix ${}_{\gamma}[\text{id}_{\mathbb{R}^3}]_{\beta}$.

Question 6

The following system of first order ODEs is given:

$$\begin{cases} f_1'(t) = f_1(t) + 2f_2(t) \\ f_2'(t) = 2f_1(t) + f_2(t) \end{cases}$$

- a) Is the given system of differential equations homogeneous or inhomogeneous?
- b) Determine the general real solution of the given system of differential equations.
- c) Determine the solution to the given system of differential equations satisfying the initial value conditions $f_1(0) = 3$ and $f_2(0) = 5$.

END OF THE EXAM