Opg 1MC

Opløsning efter 1 søjle

$$\det(\underline{\underline{A}}) = -(-1)\det\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix} + 1\cdot\det\begin{bmatrix}1 & 2\\ 4 & 1\end{bmatrix} = -7 \quad (1e)$$

Opg 2MC

$$3f''(t) - 6f'(t) + 15f(t) = 0 \iff f''(t) - 2f'(t) + 5f(t) = 0$$

karakteristisk polynomium.

$p(\lambda) = \lambda^2 - 2\lambda + 5$ $\qquad$ $p(\lambda) = 0: \ \lambda^2 - 2\lambda + 5 = 0$

$\lambda_1 = 1+2i, \ \lambda_2 = 1-2i$

$$f(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t \quad (2b)$$

Opg 3MC

$$
\begin{array}{l}
z+1 \,|\, z^3 - 2z^2 - z + 7 \,|\, \underline{z^2 - 3z} + 2 \\
\quad\ \ z^3 + z^2 \\
\hline
\quad\ \ -3z^2 - z \\
\quad\ \ -3z^2 - 3z \\
\hline
\qquad\quad 2z + 7 \\
\qquad\quad 2z + 2 \\
\hline
\qquad\qquad 5 \quad (3d)
\end{array}
$$

Alternativt:

$p(z) = d(z)\cdot q(z) + r(z)$ $\qquad$ $d(z) = z+1$

$p(-1) = 0\cdot q(z) + r(z) = 5$

Opg 4MC

4a) Hverken surjektiv eller injektiv
4b) Ikke injektiv
4c) Ikke surjektiv
4d) Ikke injektiv
4e) Ikke injektiv
4f) Bijektiv (korrekt)

Opg 5MC

Egenvektorer:

$$\begin{bmatrix} 3-i & -2 \\ 5 & -3-i \end{bmatrix} \to \begin{bmatrix} 3-i & -2 \\ 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= t \\ x_2 &= \frac{(3-i)}{2} \cdot t \end{aligned}$$

$$\underline{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3-i \end{pmatrix} \ (5b)$$

Opg 6MC

Løsningen.

Egenværdier for $\underline{A}$, $\lambda_1 = -1$, $\lambda_2 = -2$

Tilhørende egenvektorer: $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-2t}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} = \begin{bmatrix} c_1 - 2c_2 \\ c_2 \end{bmatrix} \Rightarrow \begin{aligned} c_2 &= 4 \\ c_1 &= 11 \end{aligned}$$

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 11e^{-t} - 8e^{-2t} \\ 4e^{-2t} \end{bmatrix} \ (6d)$$

Opg 7MC

$$f(1) = 0, \quad f(2) = 1, \quad f(3) = 2f(2) + (f(1))^2 = 2$$

$$f(4) = 2f(3) + (f(2))^2 = 5$$

$$f(5) = 2f(4) + (f(3))^2 = 14 \ (7c)$$

Opg 8MC

$$[L(1+z+z^2)]_\gamma = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$L(1+z+z^2) = 0 \cdot 1 + 4 \cdot z = 4z \ (8d)$$

Opg 9

$$\frac{z_2}{z_1} = \frac{(4+5i)(-2-2i)}{(-2+2i)(-2-2i)} = \frac{-8+10+(-8-10)i}{8} = \underline{\underline{\frac{1}{4} - \frac{9}{4}i}}$$

$$z_1 = \sqrt{8} \cdot e^{i\frac{3\pi}{4}}$$

$$(z_1)^4 = 64\, e^{i\left(\frac{3\pi}{4}\right)\cdot 4} = 64 e^{i3\pi} = \underline{\underline{64 e^{i\pi}}}$$

Opg 10 Total matricen:

$$\underline{\underline{T}} = \left[\begin{array}{cccc|c} 3 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow[\text{rref}]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$x_4 = t$, $x_3 = 1+t$, $x_2 = -2+t$, $x_1 = 1-t$

$$\underline{\underline{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}} , \quad t \in \mathbb{R}$$

Opg 11

a) Sandhedstabel:

P	Q	$\neg Q$	$P \Leftrightarrow \neg Q$	$(P \Leftrightarrow \neg Q) \wedge P$	$P \wedge (\neg Q)$
T	T	F	F	F	F
F	T	F	T	F	F
T	F	T	T	T	T
F	F	T	F	F	F

De to udsagn er logisk ækvivalente

b) $-2 \in \mathbb{R} \wedge -2 \notin \mathbb{N} \Rightarrow -2 \in \mathbb{R} \setminus \mathbb{N}$

$-2 \in \mathbb{Z}$

$\Downarrow$

$-2 \in S = (\mathbb{R} \setminus \mathbb{N}) \cap \mathbb{Z}$

Opg 12

$$\underline{\underline{M}} = \begin{bmatrix} | & | & | & | \\ \underline{u} & \underline{v} & \underline{w} & \underline{x} \\ | & | & | & | \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 1 & 2 & -1 & 1 \\ -1 & -1 & -1 & 4 \\ 2 & 1 & 4 & 3 \end{array}\right] \xrightarrow[\text{rref}]{} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

a) En ordnet basis for $\text{span}_{\mathbb{R}}(\underline{u}, \underline{v}, \underline{w})$ kunne væn $(\underline{u}, \underline{v})$

b) Det ses at $\underline{x}$ ikke ligger i $\text{span}_{\mathbb{R}}(\underline{u}, \underline{v}, \underline{w})$

Opg 13

Vi har $\underline{\underline{D}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\underline{\underline{Q}} = \begin{bmatrix} | & | \\ \underline{v}_1 & \underline{v}_2 \\ | & | \end{bmatrix}$

hvor λ_i er egenværdi for $\underline{\underline{A}}$ med egenvektor $\underline{v}_i$

Egenværdier findes:

Kan aflæses direkte $\lambda_1 = 1$, $\lambda_2 = 3$.

Egenvektorer findes:

$\underline{v}_1$: $\begin{bmatrix} 1-1 & 0 \\ 1 & 3-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ $\quad \underline{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\underline{v}_2$: $\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underline{\underline{D}} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\underline{\underline{Q}} = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$

Opg 14

a) ${}_{\varepsilon}[id_{\mathbb{R}^3}]_{\beta}$ kan opskrives direkte:

$${}_{\varepsilon}[id_{\mathbb{R}^3}]_{\beta} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

${}_{\beta}[id_{\mathbb{R}^3}]_{\varepsilon} = {}_{\varepsilon}[id_{\mathbb{R}^3}]_{\beta}^{-1}$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$${}_{\beta}[id_{\mathbb{R}^3}]_{\varepsilon} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b)

$${}_{\varepsilon}[L]_{\varepsilon} = {}_{\varepsilon}[id_{\mathbb{R}^3}]_{\beta}\ {}_{\beta}[L]_{\beta}\ {}_{\beta}[id_{\mathbb{R}^3}]_{\varepsilon}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -3 & 3 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ -4 & 5 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Opg 15.

Bruger sæt 3.5.1 i lærebogen

1. Basis trin. $(n=3)$

$$f(3) = 2f(2) + 3 - 1 = 2(2\cdot 1 + 2 - 1) + 2 = 8 \geq 2^3 \quad \text{ok}$$

2. Induktions trin! $(n > 3)$

Antag sand $n-1$ så vises sand for n.

Der gælder pr. antagelse $f(n-1) \geq 2^{n-1}$

$$f(n) = 2 \cdot f(n-1) + n - 1 \geq 2 \cdot 2^{n-1} + n - 1$$
$$= 2^n + n - 1 \geq 2^n \quad (\text{da } n > 3)$$

Vist $f(n) \geq 2^n$.

Sætning 3.5.1 giver at sætningen er sand for alle $n \in \mathbb{Z}_{\geq 3}$.